RESEARCH ARTICLE

Brauer's Fixed Point Theorem

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ABSTRACT: The primary goal of the paper is to deliver a simple proof of equivalence between Brouwer's fixed point theorem and the existence of equilibrium in a simple exchange model with monotonic consumers. To achieve this end, we discuss some equivalent formulations of Brouwer's theorem and prove additional ones, that are 'approximating' in character or seem to be better suited for economic applications than the standard results.

KEYWORDS: fixed point theory, equilibrium, Brouwer's theorem

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1. INTRODUCTION

One of the most useful theorems in mathematics is the incredible topological result known as Bore's fixed point theorem [1,2].

It is known that many areas of mathematics rely on, use knowledge and intertwine with other mathematical disciplines. Thus, the "fixed point" theorems have very significant applications in various mathematical disciplines [3]. Let us mention, for example, Banach's theorem on the fixed point for contractions in metric spaces, which has a significant example in proving the existence and uniqueness of solutions of ordinary differential equations. In this paper, we will talk about Brauer's no less well-known fixed point theorem.

The simplest case of this theorem is the following statement: the interval $[a, b]$ and the continuous function f are given: $[a, b] \rightarrow [a, b]$. Then the function f has at least one fixed point.

To prove this we create a new (auxiliary) function Φ defined as follows $\Phi(x) = f(x) - x$. Obviously $f(a) \geq a$ and $f(b) \leq b$ hold, respectively

$$\Phi(a) = f(a) - a \geq 0,$$

$$\Phi(b) = f(b) - b \leq 0.$$

Then, according to Bolcan's theorem, there exists a point x * such that $\Phi(x^*) = 0$, or $f(x^*) = x^*$, which proves the claim.

We will present the proof of the corresponding statement for the disk in the plane, as well as a sketch of how that statement can be transferred to higher dimensions.

2. CONTINUOUS MAPPING AND HOMEOMORPHISM

Homeomorphism or topological isomorphism is in the mathematical field of topology a special case of isomorphism between topological spaces in which topological properties apply. The two spaces between which there is a homeomorphism are called homeomorphic, from the topological point of view they are the same [4,5].

Roughly speaking, a topological space can be represented as a geometric object, and homeomorphism is the continuous mapping and distortion of that object to a new shape. Thus, for example, a square and a circle are homeomorphic in the Euclidean plane.

Let X and Y be subsets of the metric space R^n. We say that the sets X and Y are homeomorphic if there is a bijection h: $X \rightarrow Y$, such that both mappings h and h^{-1} are continuous. Such a mapping x is called a homeomorphism.

We say that the set X has the property of a fixed point, if every continuous mapping $f: X \rightarrow X$ has at least one fixed point. For example, according to what we have presented in the introduction, the interval $[a, b]$ has the property of a fixed point [6].

The fixed point property is transferred from the set X to any homeomorphic set to it. Indeed, let X have the property of a fixed point, let $x: X \rightarrow Y$ be a homeomorphism, and let $f : Y \rightarrow Y$ be an arbitrary continuous mapping. Then $h^{-1\circ} f h: X \rightarrow X$ is a continuous mapping and has at least one fixed point x. This means $x^{-1}(f(x(x))) = x$, or $f(x(x)) = x(x)$, so $y = x(x)$ is a fixed mapping point f.

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3. BRAUER'S THEORY IN THE PLANE

In this section, we will prove one generalization of the claims from the introduction. Namely, instead of an interval, we will observe the disk in a plane.

So, let a circular disk be given in a plane, that is, an area consisting of the inside of the circle together with the edge. That disk, that is, its points, will be affected by some continuous transformation, which does not have to be a bijection, but in such a way that during that transformation all the points of the disk remain inside it, arranged differently [7].

An example is a thin rubber disk, and the transformation in question can be any of its deformations in which all the points of the initial disk will remain within the newly acquired character. So we can bend, twist, crease, stretch and the like. Or, as another example, take a glass of water and use a spoon to mix the contents of the glass, but so that the particles from the surface, after mixing, remain on the surface, but in some other place [8].

Brauer's theorem says the following: every such transformation leaves at least one point immobile, that is, there is at least one point whose position will remain unchanged after the transformation. To prove this, we will assume the opposite - no point after the transformation remained in the same position, but each point became a different one.

We assign the vector *PP*' to each point *P* from the starting disk, where *P'* is the image of the point *P* at a given transformation. Exactly one vector is assigned to each point. The question is, what about the dots on the edge? Since we have defined the transformation in this way, the points from the edge of the disk will be painted inside it. Notice a point P1 from the edge, which travels in a circle counterclockwise. The direction of the vector will change, because the points on the edge have assigned vectors that are differently directed.

On the path in the circle from P_1 to P_1, the vector rotates and returns to the initial position. The number of complete revolutions made by a vector is called the "index" of the vector on the circle. It is the sum of various changes in the angle of the vector, so we take the clockwise rotation as negative, while we will take the clockwise rotation as positive. So, the index can be 0, ± 1, ± 2, ± 3,..., and it will be realized at the total change of the angle by 0, ± 360, ± 720, ± 1080, ... degrees.

We now claim that the index is equal to exactly one, that is, that the vector is one full circle counterclockwise. Remember that the vector of each point from the edge is directed towards the inside of the disk and not along the tangent. If the transformation vector is inverted for a total angle that differs from the total angle for which the tangent vector rotates (which is 360 degrees, because the tangent vector makes one complete positive rotation), the difference between the total angles through which the tangent and transformation vector rotate is some a non-zero multiplier of 360 degrees, because each makes one integer speed.

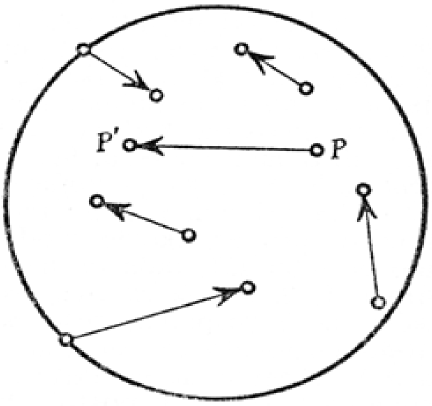

Figure 1. Transformation vectors

Thus, the transformation vector must be completely rotated at least once during the cycle from P_1 to P_1. The tangent and transformation vectors move continuously, so at some point on the circle the transformation vector will be directed in the direction of the tangent, and this is not possible.

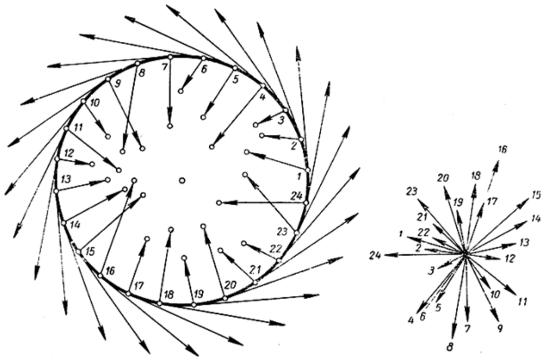

Figure 2. On the proof of Brouwer's theorem

We notice a circle that is concentric with the edge of the disk and is contained in it, together with the corresponding transformation vectors on that circle. Then the index of transformation vectors on it must also be one. If we move continuously from the edge point to any concentric circle, the index is constantly changing, but it must be an integer value, so it must be constantly equal to its initial value one, because jumping from one to another integer would be a break in the behavior of the index.

Therefore, we will notice a concentric circle that is arbitrarily small, for which the index of the transformation vector is equal to one. This is not possible because we assumed the continuity of the transformation, so the vectors on a sufficiently small circle will be approximately directed in the same direction as the vector in the center of the circle. Thus, the total change in angles can be arbitrarily small, for example less than 10 degrees for a sufficiently small circle. As the index must be an integer - it will be zero. We have reached a contradiction, which means that the assumption of the non-existence of a fixed point is not correct, which proves the theorem [9].

The proved theorem is valid both for a disk and for a triangular or square surface, or some other disk homeomorphic surface, which we see based on the considerations from the second section.

4. FIXED POINT THEOREM

Fixed point theory is a fascinating subject with a huge number of applications in various fields of mathematics. In mathematics, the fixed point theorem is a result that says that the function F will have at least one fixed point (point k for which F (k) = k), under some conditions on F that can be stated in general. Results of this kind are among the most useful in mathematics.

The fixed point theorem does not only apply to a circular disk, but, of course, to a triangle, a square, and any other figure into which a disk can be transformed by topological transformation. Indeed, if some figure A, obtained from a circular disk by this type of transformation, could be transformed into itself without fixed points, then the topological transformation of the circular disk into itself without fixed points would also be determined, and this, as we have seen, is impossible. The theorem is also generalized to the case of three-dimensional figures - spheres or cubes, but the proof here is no longer so simple [10].

In mathematics, a fixed point (sometimes shortened to fix point, also known as an invariant point) of a function is an element of the function's domain that is mapped to itself by the function. That is to say, *c* is a fixed point of the function *f* if $f(c) = c$. This means $f(f(\ldots f(c)\ldots)) = f^n(c) = c$, an important terminating consideration when recursively computing *f*. A set of fixed points is sometimes called a *fixed set*, fixed point (mathematics) [11].

Fixed-point theorems such as the above provide a powerful method for proving many "existence theorems" in various fields of mathematics, and the geometric nature of these theorems is often far from obvious. An exceptional example is Poincare's theorem, which he presented without proof just before his death in 1912. This theorem directly implies the existence of an innumerable set of periodic orbits in a limited three-body problem. Poincaré could not substantiate his assumptions, and the proof for this exceptional fact a year later was given to the American mathematician GD Birkhoff. Since then, topological methods have been applied several times and with great success in the study of the qualitative behavior of dynamical systems [12].

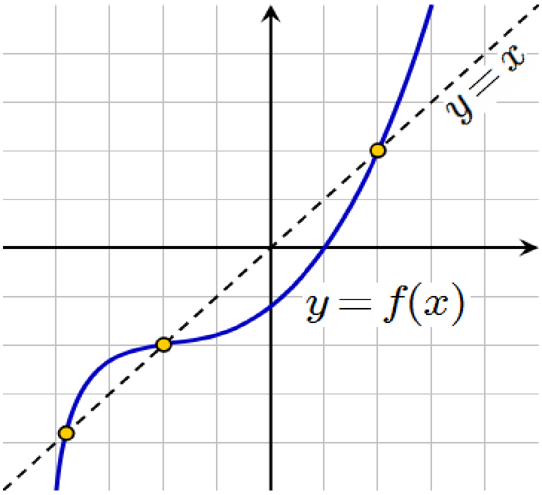

Figure 3. A function with three fixed points

Fixed point theory examines the existence of the point x belonging to the domain of function f for which stands that f(x) = x, i.e. function values are equivalent to identical function mapping. In Figure 1 three intersections of function f(x) and function y = x represent the fixed points. A more subtle analysis would lead to the conclusion that a marginal change in the f(x) function causes additional fixed points to emerge.

5. PROOF OF THE FIXED POINT THEOREM

Fixed-point theorem, any of the various mathematical theorems dealing with the transformation of points of a set into points of the same set, where it can be proved that atleast one point remains fixed. For example, if each real number is squared, the numbers zero and one remain fixed; since the transformation by which each number is increased by one does not leave a fixed number. The first example, a transformation consisting of the separation of each number, when applied to an open interval of numbers greater than zero and less than one (0,1), also has no fixed point. However, the situation changes for a closed interval [0,1], with endpoints included. A continuous transformation is one in which adjacent points are transformed into other adjacent points. Brouwer's fixed point theorem says that any continuous transformation of a closed disk (including a boundary) leaves at least one point fixed in itself. The theorem also holds for continuous transformations of points on a closed interval, in a closed sphere, or in abstract sets of higher dimensions of analogous spheres.

5.1. Fixed point theorems in phase metric spaces

The fixed point theory in indeterminate systems can be observed in different ways, and one of them is by applying phase logic. According from which segment of the problem is observed with some kind of uncertainty, the appropriate structure of the space in which the problem is considered must be used. If the distance between the elements is imprecise, then it is included in the metric uncertainty, as in the definition of the metric space phase of O. Kalev and Seikkala [13].

This model has many similarities with the probabilistic (Menger) metric spaces. Techniques, methods, and results from a fixed point in these two structures are related. Some of the results on the fixed point in the metric space phase can be found in the following papers: ([14]- [35]). Sehgal and Bharucha-Reid [21] were the first to define the generalization phase of Banach contractual condition, using the following condition:

(5.1) $M(f(x), f(y), kt) \geq M(x, y, t)$,

for everyone $x, y \in X$, $t > 0$ i $k \in (0, 1)$,
wherein $f : X \to X$.

This was followed by the generalization of the given condition for unambiguous and ambiguous mappings in the phase of metric spaces.

In the work of Shen [19], the notion of the variable distance function u was introduced phase metric spaces (X, M, T). By applying the contractual condition

(5.2) $\phi(M(f(x), f(y), t)) \leq k(t) \cdot \phi(M(x, y, t))$, $x, y \in X$, $x 6= y$, $t > 0$,

the existence of a single fixed mapping point has been proved $f: X \to X$.

6. CONCLUSION

The aim of this paper is to present the basic concepts related to fixed point theory and to list some possible applications. Brouwer's fixed point theorem says that a continuous function from a compact convex set has a fixed point. The mapping has at least one point that has remained unchanged. Keep in mind that convexity is essential. For example, the fixed point property is not true (and the theorem is inapplicable) for a function that maps the perimeter of a circle to itself. In fact, all assumptions are typical of well-constructed mathematical results. The fixed point property will not be applied to an interrupt function or to an open or unlimited set. This paper practically represents the basis for future research that would refer to determining the equilibrium competitive relationship in different markets.

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